

THE LOGIC OF QUANTUM MECHANICS – TAKE II

BOB COECKE

Abstract. We put forward a new take on the logic of quantum mechanics, following Schrödinger’s point of view that it is composition which makes quantum theory what it is, rather than its particular propositional structure due to the existence of superpositions, as proposed by Birkhoff and von Neumann. This gives rise to an intrinsically quantitative kind of logic, which truly deserves the name ‘logic’ in that it also models meaning in natural language, the latter being the origin of logic, that it supports automation, the most prominent practical use of logic, and that it supports probabilistic inference.

§1. The physics and the logic of quantum-ish logic. In 1932 John von Neumann formalized Quantum Mechanics in his book “Mathematische Grundlagen der Quantenmechanik”. This was effectively the official birth of the quantum mechanical formalism which until now, some 75 years later, has remained the same. Quantum theory underpins so many things in our daily lives including chemical industry, energy production and information technology, which arguably makes it the most technologically successful theory of physics ever.

However, in 1935, merely three years after the birth of his brainchild, von Neumann wrote in a letter to American mathematician Garrett Birkhoff : “I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more.” (sic)—for more details see [71].

Soon thereafter they published a paper entitled “The *Logic* of Quantum Mechanics” [13]. Their ‘quantum logic’ was casted in order-theoretic terms, very much in the spirit of the then reigning algebraic view of logic, with the distributive law being replaced with a weaker (ortho)modular law.

This resulted in a research community of *quantum logicians* [31, 48, 66, 69]. However, despite von Neumann’s reputation, and the large body of research that has been produced in the area, one does not find a trace of this activity neither in the mainstream physics, mathematics, nor logic literature. Hence, 75 years later one may want to conclude that this activity was a failure.

What went wrong?

1.1. The mathematics of it. Let us consider the *raison d’être* for the Hilbert space formalism. So why would one need all this ‘Hilbert space stuff’,

The work presented here is supported by the British Engineering and Physical Research Council (EPSRC), the US Office of Naval Research (ONR) and the Foundational Questions Institute (FQXi). The content of this paper reflects a series of seminars in 2010–2012 with as titles: “Monoidal categories as an axiomatic foundation”, “In the beginning God created tensor ... then matter ... then speech”, “How computer science helps bringing quantum mechanics to the masses”, “Selling categories to the masses”, or the actual title of this paper itself.

i.e. the continuum structure, the field structure of complex numbers, a vector space over it, inner-product structure, etc. Why? According to von Neumann, he simply used it because it happened to be ‘available’. The use of linear algebra and complex numbers in so many different scientific areas, as well as results in model theory, clearly show that quite a bit of modeling can be done using Hilbert spaces. On the other hand, we can also model any movie by means of the data stream that runs through your cables when watching it. But does this mean that these data streams make up the stuff that makes a movie? Clearly not, we should rather turn our attention to the stuff that is being taught at drama schools and directing schools. Similarly, von Neumann turned his attention to the actual physical concepts behind quantum theory, more specifically, the notion of a physical property and the structure imposed on these by the peculiar nature of quantum observation. His quantum logic gave the resulting ‘algebra of physical properties’ a privileged role. All of this leads us to ...

1.2. ... the physics of it. Birkhoff and von Neumann crafted quantum logic in order to emphasize the notion of quantum *superposition*. In terms of states of a physical system and properties of that system, superposition means that the strongest property which is true for two distinct states is also true for states other than the two given ones. In order-theoretic terms this means, representing states by the atoms of a lattice¹ of properties [67], that the join $p \vee q$ of two atoms p and q is also above other atoms. From this it easily follows that the distributive law² breaks down: given atom³ $r \neq p, q$ with $r < p \vee q$ we have $r \wedge (p \vee q) = r$ while $(r \wedge p) \vee (r \wedge q) = 0 \vee 0 = 0$. Birkhoff and von Neumann as well as many others believed that understanding the deep structure of superposition is the key to obtaining a better understanding of quantum theory as a whole. But as already mentioned, 75 years later quantum logic did not break through.

The Achilles’ heel of quantum logic is the fact that it fails to elegantly capture ‘composition of quantum systems’, that is, how do we describe multiple quantum systems given that we know how to describe the individual quantum systems. On the other hand, also in 1935, Schrödinger pushed forward the idea that the stuff which truly characterizes quantum behavior is precisely the manner in which quantum systems compose [72]. Over the past 30 years or so we have seen ample evidence for this claim. So-called ‘quantum non-locality’ was experimentally confirmed, and the focus on quantum information processing has revealed a wide range of quantum phenomena which all crucially depend on the manner in which quantum systems compose, most notably exponential quantum computational speed-up which led to the quantum computing paradigm [77].

Now reversing the roles, rather than explaining all of quantum theory in terms of superposition, can we maybe explain all of quantum theory in term of the manner in which quantum systems compose, including superposition?

1.3. The game plan. Here is the list of tasks we’ve set ourselves:

¹I.e. a partially ordered set with a minimal element 0 and maximal element 1, and in which each pair of elements has a supremum and an infimum. In fact, there are physical reasons for assuming that this lattice is *complete* [67, 69], i.e. arbitrary suprema and infima exist.

²*Distributivity* means that for any elements a, b, c of the lattice we have that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

³An *atom* is an element $p \neq 0$ which is such that whenever $a < p$ then $a = 0$.

- **Task 0.** First we want to solve:

$$\frac{\text{tensor product structure}}{\text{the other Hilbert space stuff}} = ???$$

that is, we want to know what remains of the Hilbert space formalism if we ‘remove all of its structure except for the manner in which systems compose’. In other words, we want to axiomatize composition of systems, which we denote by \otimes , without any reference to underlying spaces.

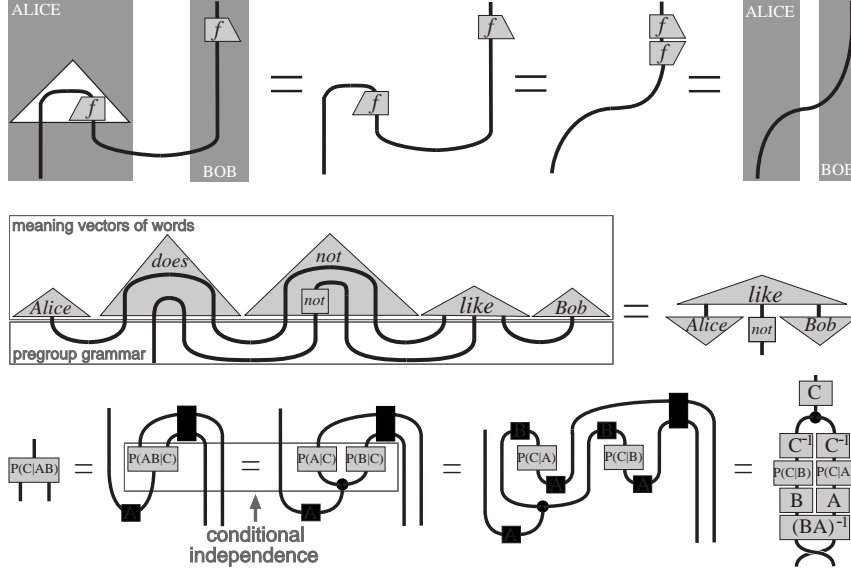
- **Task 1.** Next we investigate which additional assumptions on \otimes are needed in order to deduce experimentally observed phenomena? That is, given that the structure deduced in Task 0 applies to a wide range of theories (as we shall see below in Section 2) what extra structure do we need to add such that the resulting framework allows us to derive typical quantum behaviors.
- **Task 2.** Once this ‘typically quantum’ structure has been identified, we take on the challenge to find this same structure elsewhere in what we usually conceive as ‘our classical reality’. This may involve looking at this classical reality through a ‘novel pair of glasses’.

And, ... here are the resulting outcomes:

- **Outcome 0:** That was an easy one. The solution to this has been around for quite a while. It is called *symmetric monoidal category* [11]. In fact, as discussed in [24, 33], physical processes themselves form a *strict* symmetric monoidal category, while set theory based models such as the Hilbert space model are typically non-strict, which invokes so-called ‘coherence conditions’ [65] between ‘natural transformations’ [46]. But one can show that an arbitrary symmetric monoidal category is always ‘categorically equivalent’ to a strict symmetric monoidal category, which means that, up to isomorphisms, whatever one can do with a non-strict one, one can do with a strict one too. Hence, here we will only spell out strict symmetric monoidal categories, in terms of their *graphical language* [57, 68], that is, a language which is such that an equational statement holds in it if and only if it follows from the axioms of a strict symmetric monoidal category.
- **Outcome 1a:** Quoting Princeton philosopher Hans Halvorson in his editorial to the volume *Deep Beauty: Understanding the Quantum World through Mathematical Innovation* which marked 75 years since the publications of von Neumann’s quantum formalism [52]: “What is perhaps most striking about Coecke’s approach is the sheer ratio of results to assumptions.” As we shall see below, with very little additional structure one can already derive a wide range of quantum phenomena, and the required computations are utterly trivial. This is in sharp contrast with Birkhoff-von Neumann quantum logic where one couldn’t derive much; and in the case that one could derive something physically relevant one had to work really hard.
- **Outcome 1b:** Moreover, exposing this structure has already helped to solve standing open problems in quantum information, e.g. [14, 45, 55], and provided novel insights in the nature of quantum non-locality [27, 28].
- **Outcome 1c:** The diagrammatic framework underpinning strict symmetric monoidal categories has meanwhile been adopted by several leading

researchers in quantum foundations e.g. [17, 18, 53]; quoting Lucien Hardy in [53]: “[...] we join the *quantum picturalism* revolution [24]”.

- **Outcome 2a:** Observe the following similar looking pictures:



These are respectively taken from a physics paper on the flow of information in quantum protocols [1, 21, 24], a linguistics paper on how to compute the meaning of a sentence given the meaning of its words [20, 38], and a probability theory paper that axiomatizes Bayesian inference [39]. The graphical calculi are in each case very similar, which points at a common reasoning systems in each of these very distinct areas. Note in particular that in each case the data of interest is of a fundamentally quantitative nature. Could this be pointing at the existence of some sort of *quantitative logic*, which is not typical to these areas but of a more universal nature?

So let us now consider ...

1.4. ... the logic of it. What is logic? The previous century has known a huge proliferation of logics of various kinds, and there probably are as many opinions of what logic actually is. Rather than making a case for one or another logical paradigm we will take a pragmatic stance and conceive logic in terms of its origin and its most prominent practical use:

- **Origin: structure in natural language.** The origin of logic, tracing back to Aristotle, is that it is about ‘arguments in natural language’. Consider for example the sentence: “Alice and Bob either ate everything or nothing, then got sick.” By using connectives, quantifiers, variable f referring to food, constants a (lice) and b (ob), and predicates $Sick(\text{person})$ and $Eat(\text{person}, \text{some kind of food})$ we can formalize this as follows:

$$(\forall f : Eat(a, f) \wedge Eat(b, f)) \vee \neg(\exists f : Eat(a, f) \wedge Eat(b, f)) \Rightarrow Sick(a), Sick(b)$$

However, statements like this are still tightly related to a truth-concept, that is, we classify statements in terms of these either being true or not.



FIGURE 1. The theory[mine] website which allows one to buy an automatically generated theorem and name it after someone. It is a novelty gift spin-off from the automated theory exploration expertise at Edinburgh University—see [15] for the science.

Clearly there is a lot more to the *meaning* of a sentence than it either being true or false. This leads us to the following questions: What do we mean by meaning? What is the logic governing meaning, more specifically, how do meanings of words interact to form meanings of sentences?

- **Use: automated reasoning.** Logic now forms the foundation for fields like automated proof checking and automated theorem proving in computer science, which are key to modern methods for verifying the correctness of new software and hardware. Logic also controls robot behaviors in artificial intelligence. Even more adventurous is *automated theory exploration*, where one does not only try to automatically proof theorems, but also generate them, which is a much harder task (cf. P vs. NP)—see also Figure 1.

Our diagrammatic framework appeals to both of these senses of logic, and in doing so produced important new applications in each of these areas:

- The above depicted framework for modeling how meanings of words interact to form meanings of sentences, introduced by Clark, Sadrzadeh and the author in [20, 38], is the first to do so based on a clear conceptual underpinning. It was a cover heading feature in *New Scientist* [5] and meanwhile greatly improved performance of several natural language processing tasks [51]. We explain this framework in Section 4, as well as its structural relationship the graphical quantum formalism.
- The diagrammatic formalism underpins the automated reasoning software **quantomatic** developed at Oxford and Google by Dixon, Kissinger, Merry, Duncan, Soloviev and Frot—see also Figure 2. More recently, work on automated theory exploration of graphical theories also started at Oxford [59], building further on the work done at Edinburgh [56]. We won't discuss this here; details are in [41–43] and on the **quantomatic** website.

§2. Minimal process logic. By a process logic we simply mean any strict symmetric monoidal category, and by minimal we mean that at this stage we consider no structure (yet) other than the strict symmetric monoidal structure. We explain this structure in terms of its graphical language.

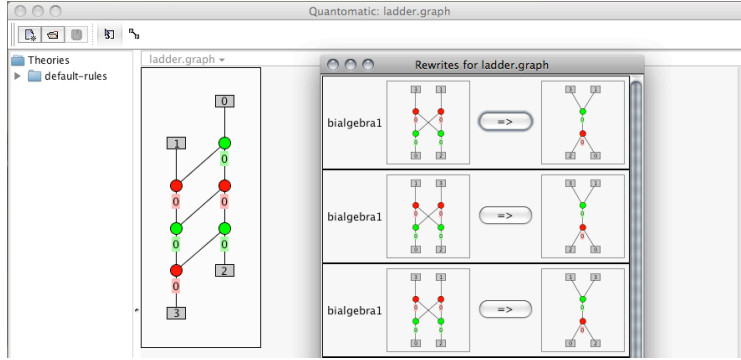
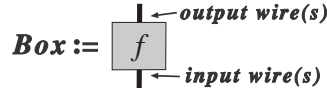


FIGURE 2. Screenshot of the quantomatic software developed in a collaboration between Oxford and Google, which can be downloaded from <http://sites.google.com/site/quantomatic/>.

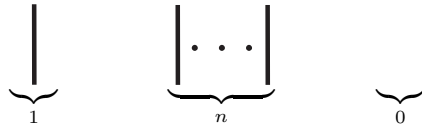
We could as well have given a symbolic presentation. We refer the reader to [24, 33] for such a symbolic presentation, exemplified for the specific case of cooking processes, and how they compose to make up recipes—[24] also discusses how a process logic explains why tigers have stripes while lions don't.

2.1. Graphical language. The data of a minimal process logic consists of processes, represented by boxes, each of which takes some type of systems as its input, represented by (an) input wire(s), and some type of system as its output:



These types may be compound, or trivial, i.e. representing ‘no system’:

one system n sub-systems no system



Examples of types could be a particular quantum system, classical data of a certain size, grammatical types, e.g. the type of a noun, verb, or a sentence, etc. A process with no input wire is called a *state*—one can think of these as ‘preparation processes’. Those with neither an input type nor an output type are called *values*. A process without an output type is called a *valuation*.

The connectives of a minimal process logic constitute composition of processes. There are two modes: *sequential* or *causal* or *connected* composition, and, *parallel* or *acausal* or *disconnected* composition, respectively depicted as:



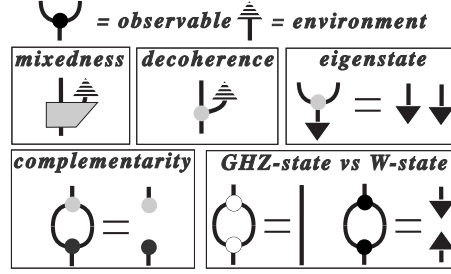
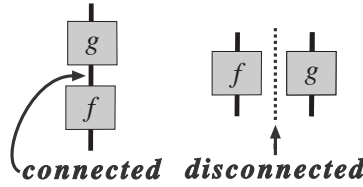


FIGURE 3. Examples of quantum mechanical concepts that can be expressed in purely topological terms, with the help of some new graphical elements. They are taken from [23, 26, 29, 35, 37].

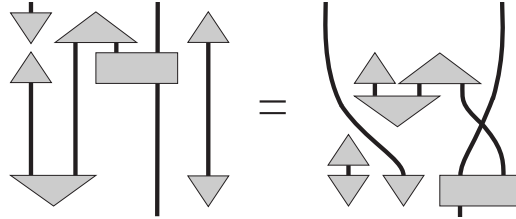
So by post-composing a state with a valuation one obtains a value. Note that sequential composition requires the output type of f to be equal to the input type of g while no such restriction exists for parallel composition.

The formal paradigm underpinning minimal process logic is a topological one:

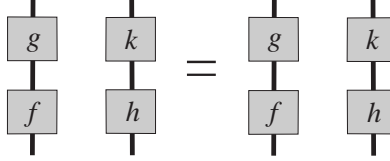


The topology captures ‘what interacts with what’, a wire standing for interaction while no wire stands for no interaction. It is surprising how much concepts can be expressed purely in these topological terms—e.g. see Figure 3 for some topologically characterized quantum mechanical concepts.

The computational content of minimal process logic boils down to the simple intuitive rule that topologically equivalent diagrams are equal. Hence computation proceeds by topological deformations:



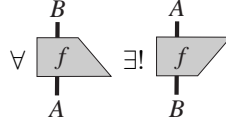
There is no additional equational content to a minimal process logic. This may sound surprising, since a strict symmetric monoidal category is subject to a number of axioms. The explanation is that in the graphical language all these equations become tautologies. For example, denoting sequential composition by \circ and parallel composition by \otimes , the ‘bifunctionality equation’ $(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$ of monoidal categories becomes:



In terms of processes this means that ‘ g after f , while, k after h ’ is the same as ‘ g while k , after, f while h ’.

§3. Quantum process logic - Take IIa. Our next goal is to derive some non-trivial quantum phenomena by endowing a minimal process logic with a tiny bit of extra structure, identified by Abramsky and the author in [1, 2].

3.1. Dagger compact structure. The first bit of extra structure will induce some kind of metric on the states, namely, we will ask that each state can be turned into a valuation; applying this valuation to any other state will yield a value. Note that this is exactly how the highly successful Dirac notation [40] works: a ket $|\psi\rangle$ can be turned into a bra $\langle\psi|$, and when composing $\langle\psi|$ with another ket $|\phi\rangle$ we obtain a bra-ket $\langle\psi|\phi\rangle$ i.e. an inner-product. Since states may themselves arise by composing processes other than states, we will allow for the inputs and the outputs of any process to be ‘flipped’:



Note that flipping twice yields the original box, so flipping is involutive, and it is also clear that it preserves parallel composition, while it reverses sequential composition. We refer to flipping as the *adjoint* or *dagger*.⁴

So far we haven’t said anything specific about the parallel composition. Now we will truly follow Schrödinger’s path and specify in which manner quantum systems compose differently than classical systems. In other words, we will assert that pure quantum states admit entanglement, diagrammatically:

$$\frac{\text{quantum}}{\text{classical}} = \frac{\begin{array}{c} \text{---} \text{ } \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}}{\begin{array}{c} \text{---} \text{ } \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}} \neq \frac{\begin{array}{cc} \text{---} & \text{---} \\ \diagdown & \diagup \\ \text{---} & \text{---} \end{array}}{\begin{array}{cc} \text{---} & \text{---} \\ \diagup & \diagdown \\ \text{---} & \text{---} \end{array}} = \frac{\begin{array}{cc} \text{---} & \text{---} \\ \diagdown & \diagup \\ \text{---} & \text{---} \end{array}}{\begin{array}{cc} \text{---} & \text{---} \\ \diagup & \diagdown \\ \text{---} & \text{---} \end{array}}$$

That is, a quantum state of two systems can in general not be described by describing the state of its parts. Note that this is also not the case for probabilistic classical data: a situation of two systems which comes with the promise that the states of the system are the same but unknown, can also not be described by independently describing the state of each system. However, in quantum theory

⁴From the perspective of Birkhoff-von Neumann quantum logic, one could conceive this as the analog to an orthocomplementation on the lattice structure. That is, an order-reversing involution. Note in particular that for non-Boolean lattices an orthocomplementation is a structure, not a property, as there can exist many different ones on the same lattice. In lattice theoretic terms the linear algebraic adjoint indeed arises as an expression involving Galois adjoints $(-)^*$ and orthocomplementation $(-)^{\prime}$, namely $f^{\dagger}(a) = (f^*(a'))^{\prime}$ [30, 47].

this already occurs for states on which there is no uncertainty, that is, for which there exists a measurement that yields a particular outcome with certainty.

So how do we provide a constructive witness for the fact that the state of two systems does not ‘disconnect’ in two separate one-system states? Simply by explicitly introducing a special two system state which is obtained by connecting its two outputs with a cup-shaped wire:



Sticking to our topological paradigm, such a cup-shaped state for example obeys:

$$(1) \quad \text{cup} \cup \text{cap} = \text{straight wire}$$

The equivalent symbolic expression for this equation would be:

$$(\cap \otimes 1) \circ (1 \otimes \cup) = 1$$

where 1 stands for a single straight wire and \cap is obtained simply by flipping \cup i.e. its adjoint. We obtain a strict *dagger-compact category* [1, 2].

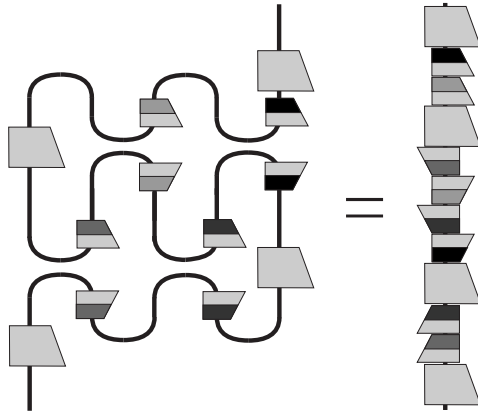
3.2. Deriving physical phenomena. We assumed the existence of an adjoint for any box and represent it via flipping. Cup- and cap-shaped wires also enable us to ‘define’ the *transpose* which we depict by rotating a box 180°:

$$\text{box } f = \text{rotated box } f$$

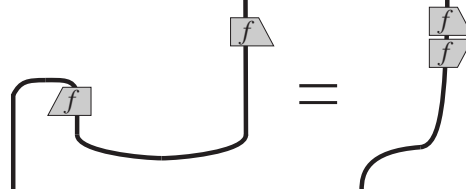
It then immediately follows that we have:

$$\text{box } f \cup \text{cap} = \text{rotated box } f \cup \text{cap} = \text{cup} \cup \text{box } f$$

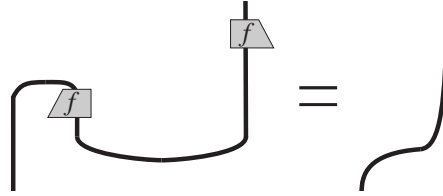
that is, we can slide boxes across cup- and cap-shaped wires. Going berserk,



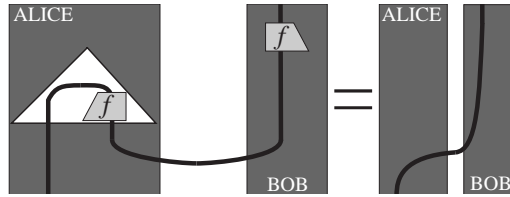
that is, we can treat the entire graphical calculus for dagger compact categories in terms of beads which slide on wires. Now for some physics. We have:



and we choose f such that its composite with its adjoint yields the identity, something to which we refer as *unitarity*. Hence:

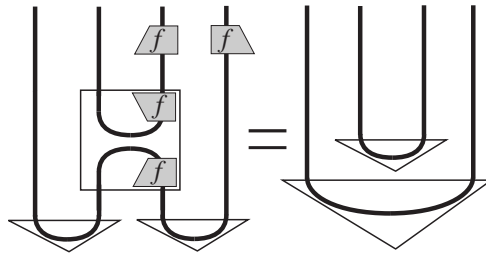


Introducing agents Alice and Bob yields *quantum teleportation*:



Note that, given that the quantum mechanical formalism was born in 1932, that this phenomenon took 60 years to be discovered [12]. The standard quantum mechanical formalism provides no indication whatsoever that something like this would be possible, so one had to rely on sheer luck to discover it.

A more detailed discussion of this graphical derivation and its physical interpretation is in [21, 24]. Similarly we derive another quantum mechanical feature, the *entanglement swapping* protocol [79]:



So how much quantum mechanics can we derive in this calculus?

3.3. Logical completeness wrt Hilbert spaces. The diagrammatic language presented above is directly related to the symbolic notion of a dagger compact category as follows:

THEOREM 1 (Kelly-Laplaza; Selinger [58, 75]). *An equational statement between expressions in the dagger compact categorical language holds if and only if it is derivable in the above described graphical calculus.*

Evidently there are many dagger compact categories, to mention two:

- Wires represent finite dimensional Hilbert spaces, boxes linear maps, the dagger is the linear algebraic adjoint, sequential composition is ordinary function composition, and the parallel composition of wires is the tensor product while parallel composition of boxes is the Kronecker product.
- Wires represent sets, boxes relations, the dagger is the relational converse, sequential composition is composition of relations, and parallel composition is the cartesian product.

The description of the compact structure for each of these as well as some more examples can be found in [33]. Evidently these two examples have very different spaces and one would evidently not associate sets and relations with quantum processes. Hence one could wonder how much one can actually derive in (the graphical calculus for) dagger compact categories. The answer is surprising.

THEOREM 2 (Hasegawa-Hofmann-Plotkin; Selinger [54, 76]). *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the dagger compact category of finite dimensional Hilbert spaces, linear maps, tensor product and linear algebraic adjoints.*

To put this in more quantum physics related terms, any equation involving:

- states, operations, effects, ...
- Bell-state, Bell-effect, transposition, conjugation, ...
- inner-product, linear-algebraic trace, Hilbert-Schmidt norm, ...
- adjoints (e.g. self-adjointness and unitarity), projections, positivity, ...
- complete positivity (cf. [75]), ...

holds in quantum theory if and only if it can be derived in the graphical language.

§4. Natural language process logic. Before continuing with the further development of quantum process logic, we turn our attention on something completely different: meaning in natural language, in particular, the from-word-meaning-to-sentence-meaning process. Meaning here manifestly goes beyond simply assigning truth values to sentences.

4.1. From word meaning to sentence meaning. Consider as given the meanings of words. This can mean many things, for example, one has a dictionary available. On the other hand, there are no dictionaries for entire sentences. So how do we know what a sentence means? There must be some kind of mechanism, used by all of us, for transforming the meaning of words into the meaning of a sentence, since surely, we all understand sentences that we may have never heard before in our lives, provided we understand all of its words.

There is a technological side to this. Search engines such as google and other natural language processing tools also have an understanding of meanings of words which they use to provide us with the most relevant outputs for our queries. The model of word meaning which these engines employ enables them to produce outputs that include words that are closely related to the words in our query, i.e. there doesn't have to be an exact match.

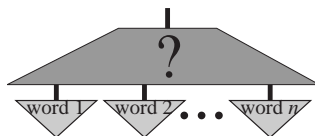
However, searching on Google for “I want something that allows me to go faster than when I only use my legs” returns among its top hits: “Difference Between Oxycontin and Oxycodone”, “What are good ways for a girl to [XXX]”, “How to

Sprint Faster: 6 steps - wikiHow', "My Story - Onelegtim.com - Retired Police Officer & [...]" and "Golf Swing Power: What Your Legs Should Be Doing [...]". Neither of these point me in the direction of appropriate vehicles that would serve my purpose, so clearly there is no understanding of the meaning of my query. The reason is the lack of a theory that produces the meaning of a sentence from the meanings of its words, whatever the manner is in which we describe the meaning of words.

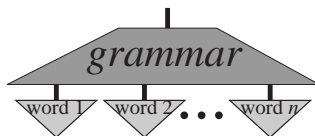
Now, representing grammatical types of words by wires and their meanings by state-boxes we can depict a string of words as:



But the overall type, i.e. the overall wire structure, depends on the grammatical structure of the sentence. However, sentences with different grammatical structure may have the same meaning, and more general, we would like to have a fixed type for the meaning of all sentences. Hence there is some process, the from-word-meaning-to-sentence-meaning process, which transforms the meanings of the string of words in the meaning of the sentence made up from these:



What drives this process? That is, given a string of words, what mediates their interaction? The answer is obvious:

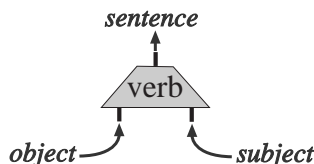


since grammatically incorrect sentences have no clear meaning anyway.

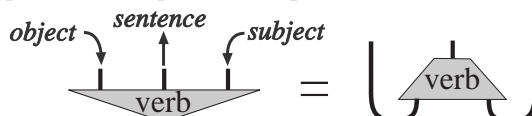
We can now describe the problem for from-word-meaning-to-sentence-meaning processes in more precise terms:

- Given a theory of word meaning, and given a theory of grammar, how can we combine these into an algorithm which produces the meaning of sentences from the meanings of its words?

As already mentioned, this problem was addressed by Clark, Sadrzadeh and the author in [20, 38]. Let's stay at an abstract level a bit longer, before we will describe concrete theories of word meaning and grammar. What is a verb? A transitive verb is something that requires an object and a subject in order to yield a grammatically correct sentence. So we can think of a transitive verb as a process with three wires, two respectively requiring an object and a subject, and one producing the sentence:

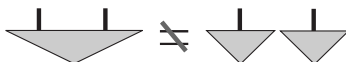


Since we rather represent a verb as a state we can use transposition, as defined above, to turn inputs into outputs and represent the verb as:



You may ask where these cups suddenly come from, but here we already anticipate the description of grammatical structure that we discuss below.

Note in particular also that for these kinds of word-states we again have:



since otherwise, for the case of a transitive verb, the meaning of the sentence would not depend on the meanings of the nouns, which could have dramatic consequences. For example, considering the verb ‘hate’, it would be sufficient for one person to hate another person in order for everyone to hate everyone.

4.1.1. A theory for word meaning. The current dominant theory of word meaning for natural language processing tasks is the so-called *distributional* or *vector space model* of meaning [73]. It takes inspiration from Wittgensteins philosophy of meaning is use [78], whereby meanings of words can be determined from their context, and works as follows. One fixes a collection of n words, the *context words*, and considers an n dimensional vector space with chosen basis where each basis vector represents one of the context words. Then one selects a huge body of written text, the *corpus*. E.g. the internet, all editions of a certain newspaper, all novels, the British National Corpus⁵ which is a 100 million word collection of samples of written and spoken language from a wide range of sources, etc. Next one decides on a *scope*, that is, a small integer k , and for each context word x one counts how many times $N_x(a)$ a word a to which one wants assign a meaning occurs at a distance of at most k words from x . One obtains a vector $(N_1(a), \dots, N_n(a))$, which one normalizes in order to obtain $(\phi_1(a), \dots, \phi_n(a))$, the *meaning vector* of a . Now, in order to compare meanings of words, in particular, how closely their meanings are related, one can simply compute the inner-product of their meaning vectors.

4.1.2. A theory of grammar. Algebraic gadgets that govern grammatical types have been around for quite a bit longer [4, 10, 19, 62]. There are several variants available, each with their pro’s and con’s; here we will focus on Lambek’s *pregroups* [63]. Philosophically, these algebraic gadgets trace back to Freges principle that the meaning of a sentence is a function of the meaning of its parts [49]. However, this is only manifest in that these algebras all have a composition operation that allows to build larger strings of words from smaller strings of words.

⁵This can be accessed at <http://www.natcorp.ox.ac.uk/>

These algebras also have a relation \leq where $a \cdot \dots \cdot z \leq t$ means that the string of types $a \dots z$ has as its overall type t . For example, $n \cdot tv \cdot n \leq s$ expresses the fact that a noun, a transitive verb and a noun make up a sentence s . Finally, there are additional operations subject to certain laws which make up the actual structure of the algebra, and these would allow one to derive correct statements such as $n \cdot tv \cdot n \leq s$.

For the specific case of pregroups, these additional operations are a *left inverse* $^{-1}(-)$ and a *right inverse* $(-)^{-1}$, subject to $x \cdot ^{-1}(x) \leq 1$ and $(x)^{-1} \cdot x \leq 1$ where 1 is the unit for the composition operation, as well as to $1 \leq ^{-1}(x) \cdot x$ and $1 \leq x \cdot (x)^{-1}$. Now we have to assign grammatical types to the elements of a pregroup. Some will be atomic, i.e. indecomposable, while others like transitive verbs will be assigned compound types. Concretely, $tv = ^{-1}(n) \cdot s \cdot (n)^{-1}$, hence

$$n \cdot tv \cdot n = n \cdot (^{-1}(n) \cdot s \cdot (n)^{-1}) \cdot n = (n \cdot ^{-1}(n)) \cdot s \cdot ((n)^{-1} \cdot n) \leq 1 \cdot s \cdot 1 = s,$$

so the string of types ‘noun transitive verb noun’ indeed makes up a grammatically correct sentence. We can depict this computation graphically as follows. We start with five systems of respective types n , $^{-1}(n)$, s , $(n)^{-1}$ and n :

$$\begin{array}{ccccc} | & | & | & | & | \\ \mathbf{n} & ^{-1}(\mathbf{n}) & s & (\mathbf{n})^{-1} & \mathbf{n} \end{array}$$

Then, we use caps to indicate that n and $^{-1}(n)$, and, $(n)^{-1}$ and n , cancel out:

$$\begin{array}{ccccc} \cap & & | & & \cap \\ \mathbf{n} & ^{-1}(\mathbf{n}) & s & (\mathbf{n})^{-1} & \mathbf{n} \end{array}$$

so that at the end the only remaining system is the sentence type. The caps here represent the equations $x \cdot ^{-1}(x) \leq 1$ and $(x)^{-1} \cdot x \leq 1$. In fact, this is not just an analogy with the graphical language of compact categories. Pregroups *are* in fact compact categories! To see this, any partial order is a category, the composition provides the tensor, and while equations $x \cdot ^{-1}(x) \leq 1$ and $(x)^{-1} \cdot x \leq 1$ provide caps, equations $1 \leq ^{-1}(x) \cdot x$ and $1 \leq x \cdot (x)^{-1}$ provide cups. More details on this are in [38]. The reason that there are two kinds of caps and cups is the fact that we are not allowed to change the order of words in a sentence while two physical systems do not come with some ordering. In category-theoretic terms, here we are dealing with a *non-symmetric tensor*.

4.2. Combining theories. The structural similarity between the pregroup theory of grammar, and the vector spaces for word meaning when organized as a dagger compact category, is exactly what we will exploit to explicitly construct the from-word-meaning-to-sentence-meaning process. We consider the graphical representation of the proof of grammatical correctness of a sentence, substitute the sentence types by meaning vectors of the particular words we are interested in, and substitute the caps by the vector space caps, so we obtain:



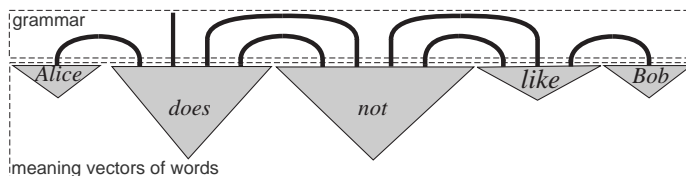
where the dotted line indicates the linear map that when applied to the vector $\overrightarrow{Alice} \otimes \overrightarrow{hates} \otimes \overrightarrow{Bob}$ produces the vector that we take to be the meaning of a

sentence. By rewriting this using transposition, as in Section 4.1, the verb now acts as a fuction on the object and the subject:

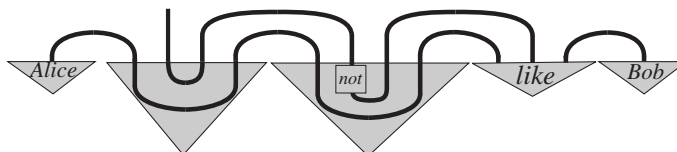


The meanings of all sentences live in the same vector space so we can again simply use the inner-product to measure their similarity. Grefenstette and Sadrzadeh have recently exploited this theory for standard natural processing tasks and their method outperforms all existing ones [51].

What about the cups? They can be used to model ‘special words’ like “does” and “not”, which have a clear ‘logical’ meaning. Here is an example of this:



As above, the wire structure here is obtained from the types of these words according to the pregroup grammar. Using cups we can model the meaning of ‘does’, that is, ‘does nothing really’, and ‘not’, that is, ‘negates meaning’, for which we use an input-output *not*-box that does just that:



and then we can simply use homotopy to compute:



which is exactly what we would expect the meaning of Alice doesn’t liking Bob to be: the negation of Alice liking Bob. This example also shows how the wires are mediating the ‘flow’ of word meaning in sentences. They allow for the words Alice and like, while far apart in the sentence, to interact.

Turning things upside-down, one can now ask the question: why are there algebraic gadgets that describe grammatical correctness, i.e. why do these even exist. Our theory of word meaning explains this: they witness the manner of how word meanings interact to form the meaning of a sentence.

4.3. An aside: quantizing grammar. An interesting analogy arises, which was first observed by Louis Crane, and which is discussed in detail in [70]. An important area of contemporary mathematics is the study of Topological Quantum Field Theory (TQFT) [6–8]. While it takes its inspiration from quantum field theory, it has become an area of research in its own right, mainly within

topology. The object of study is a monoidal functor:

$$F : n\mathbf{Cob} \rightarrow \mathbf{FVect}_{\mathbb{K}} :: \text{[diagram of a pair of pants]} \mapsto V$$

from the compact category of closed $(n - 1)$ -dimensional manifolds with diffeomorphism classes of n -dimensional manifolds connecting the closed $(n - 1)$ -dimensional manifolds as morphisms, to the compact category of vector spaces over some field \mathbb{K} .⁶ Now, rather than taking a category of topological structures as domain, we can take a pregroup as domain, i.e. a category of grammatical structures, and obtain a *grammatical quantum field theory*:

$$F : \mathbf{Pregroup} \rightarrow \mathbf{FVect}_{\mathbb{K}_+} :: \text{[diagram of a cup and a cap separated by a vertical line]} \mapsto V$$

§5. Quantum process logic - Take IIb. Dagger compact categories capture a substantial number of quantum mechanical concepts, and the dagger compact category **FdHilb** related to the von Neumann model described in Section 3.3, is complete with respect to them. But they are by no means universal with respect to quantum theory, by which we can mean two different things:

- that they do not capture all quantum mechanical concepts, and,
- that the language is not rich enough to describe all processes in **FdHilb**.

Examples of concepts that are not captured by dagger compact language are the classical data obtained in measurements, observables themselves, and relationships between these e.g. complementarity. Examples of **FdHilb**-processes not expressible in dagger compact language are basic quantum computational gates such as the CNOT-gate, phase-gates etc. We will now present an extended graphical language which does capture all of these. This was established in a series of papers by Pavlovic, Paquette, Duncan, and the author in [25, 26, 34, 35]. The calculus was also rich enough to address a number of concrete quantum computational and quantum foundational problems e.g. see [14, 27, 28, 45, 55].

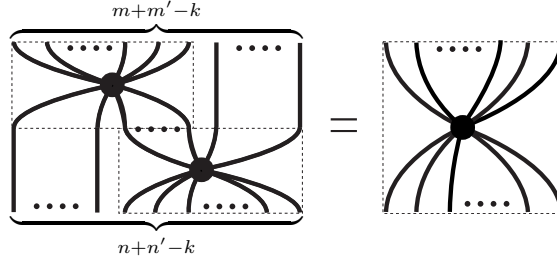
Rather than only allowing for wires we allow for ‘dots’ at which wires branch into multiple wires, or none. We refer to these dots as ...

$$\text{‘spiders’} = \left\{ \begin{array}{c} m \\ \text{[diagram of a spider with } m \text{ inputs and } n \text{ outputs]} \\ n \end{array} \right\}_{n,m}$$

So what is the analogue the topological calculus with cups and caps, and in particular, eq.(1)? Similarly to ‘however one bends a wire, it still remains just a wire that acts as an identity’, any web of spiders with the same overall number of inputs and outputs, independent of how the web is build up, is again the same.

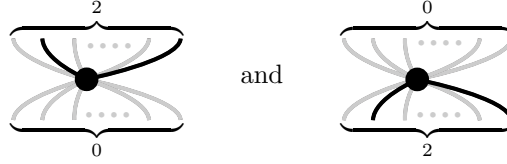
⁶If the field has a non-trivial involution then this category has a dagger too (\neq transposition).

So for any $k > 0$:

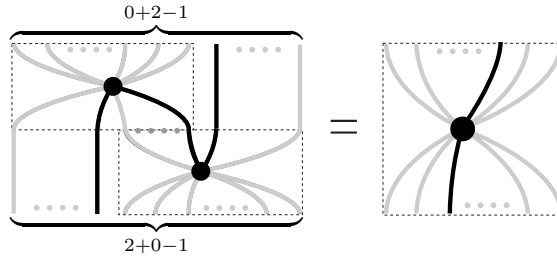


Hence, the rule governing spider calculus is that if two spiders ‘shake legs’, they fuse together. Again in other words, it only matters what is connected to what, but not the manner in which this connection is realized.

This in particular implies that for the specific spiders:



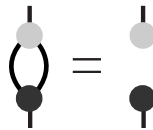
we obtain eq.(1):



so *reasoning with spiders strictly generalizes reasoning with wires*.

In **FdHilb** a family of spiders of the above kind on-the-nose captures an orthonormal basis, which is a non-trivial result. Firstly, one can show that reasoning with those spiders is equivalent to working with a so-called dagger special commutative Frobenius algebra [32, 60, 61]. Next one shows that these dagger special commutative Frobenius algebras in **FdHilb** are the same thing as orthonormal bases [36]. Since bases allow to represent observables and classical data, we almost reached our goal, except for the fact that quantum theory only becomes interesting if we consider several ‘incompatible’ bases.

So now we consider two different families of spiders, represented by a different gray scale. What happens if a dark gray and a light gray spider which represent complementary observables ‘shake legs’? Well, their ‘legs fall off’:



This was shown by Duncan and the author in [26]. Such a pair of differently colored spider families that interact in this manner forms the basis of a rich calculus with many more extra features than the ones described here. We refer the interested reader to [24, 26, 27] for more details and concrete applications.

§6. The remaining challenge. In this paper we pushed forward the idea that the diagrammatic languages describing quantum phenomena as well as meaning-related linguistic phenomena may constitute some new kind of quantitative logic. The same logic also governs Bayesian inference, Bayesian inversion boiling down to nothing but transposition for appropriately chose cups and caps:

The diagram shows an equation between two expressions. On the left, a vertical line passes through a rectangular box labeled 'AB'. On the right, a vertical line passes through a rectangular box labeled 'BA'. Above the 'BA' box, the line forms a loop that goes up, over the box, and back down to the line below the box. The two expressions are separated by an equals sign.

This was established by Spekkens and the author in [39], to which we refer for details. So where does traditional logic fit into this picture?

One perspective is to start with standard categorical logic [3, 9, 64]. The compact structure can then be seen as a resource sensitive variant (as in Linear Logic [50, 74]) which is degenerate in the sense that conjunction and disjunction coincide [22, 44]. We do not subscribe (anymore) to conceiving the diagrammatic logic as a ‘degenerate hyper-deductive variant’ of standard logic in categorical form since this does not recognize the quantitative nor the process content.

Rather, we would like to conceive the quantitative diagrammatic logic as ‘the default thing’ from which traditional qualitative logic arises via some kind of structural collapse. There are several results that could be taken as a starting point in this direction, for example, the generalization in [34] of Carboni and Walters’ axiomatization of the category of relations [16]. But since this still belongs to the world of speculation, we leave this to future writings.

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UNIVERSITY OF OXFORD,
 DEPARTMENT OF COMPUTER SCIENCE,
 QUANTUM GROUP
E-mail: coecke@cs.ox.ac.uk